On unentangled Gleason theorems for quantum information theory

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It is shown here that a strengthening of Wallach's Unentangled Gleason Theorem can be obtained by applying results of the present authors on generalised Gleason theorems for quantum multi-measures arising from investigations of quantum decoherence functionals.

I. INTRODUCTION

In an interesting recent paper Wallach [1] obtained an *unentangled* Gleason theorem. His work was motivated by fundamental problems in quantum information theory, in particular: to what extent do local operations and measurements on multipartite quantum systems suffice to guarantee the validity of a theorem of Gleason-type and thus a Born-type rule for probabilities? His positive result is formulated in terms of partially defined frame functions, defined only on the *unentangled states* of a finite product of finite dimensional Hilbert spaces. We shall show that Wallach's theorem, and also its generalisation to infinite dimensions, can readily be derived from results which were obtained by us in our investigations of generalised Gleason theorems for quantum bi-measures and multi-measures [2–6]. The physical motivation for our earlier work arose from the so-called histories approach to quantum mechanics [7,8]. Our more general approach relies on the generalised Gleason theorem obtained by Bunce and one of us [9–11].

II. PRELIMINARIES

Throughout this note $\mathcal{H}$ is a Hilbert space, $\mathcal{S}(\mathcal{H})$ is the set of unit vectors in $\mathcal{H}$, and the sets of projections, compact operators or bounded operators are denoted by $\mathcal{P}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$, or $\mathcal{B}(\mathcal{H})$ respectively.

A *quantum measure* for $\mathcal{H}$ is a map $m : \mathcal{P}(\mathcal{H}) \to \mathbb{C}$ such that $m(p+q) = m(p) + m(q)$ whenever p and q are orthogonal. If m takes only positive values and $m(1) = 1$, then m is a *quantum probability measure*. If, whenever $\{p_i\}_{i\in I}$ is a family of mutually orthogonal projections, $\sum_i m(p_i)$ is absolutely convergent and $m(\sum_i p_i) = \sum_i m(p_i)$, then m is said to be *completely additive*.

The essential content of Gleason's original theorem [12] is that if m is a positive, completely additive quantum measure on $\mathcal{P}(\mathcal{H})$, then it has a unique extension to a positive normal functional ϕ_m on $\mathcal{B}(\mathcal{H})$, whenever the Hilbert space $\mathcal{H}$ is not of dimension 2. It then follows from routine functional analysis that

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there exists a unique positive, self-adjoint trace class operator T on $\mathcal{H}$ such that $\phi_m(x) = \mathrm{Tr}(Tx)$ for each $x \in \mathcal{B}(\mathcal{H})$, i.e., $m(p) = \mathrm{Tr}(Tp)$ for each $p \in \mathcal{P}(\mathcal{H})$. As a tool to help him prove his theorem, Gleason introduced the notion of a frame function. A (positive) *frame function* for $\mathcal{H}$ is a function $f : \mathcal{S}(\mathcal{H}) \to \mathbb{R}^+$ such that there exists a real number w (the *weight* of f) such that, for any orthonormal basis of $\mathcal{H}$, $\{x_i\}_{i\in I}$, $\sum_i f(x_i) = w$. There is a bijective correspondence between frame functions for $\mathcal{H}$ and positive, completely additive quantum measures on $\mathcal{P}(\mathcal{H})$, see [12].

III. UNENTANGLED FRAME FUNCTIONS AND QUANTUM MULTI-MEASURES

Let $\mathcal{H}_1, \cdots, \mathcal{H}_n$ be Hilbert spaces. An *unentangled* element of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ is a vector which can be expressed in the form $x_1 \otimes \cdots \otimes x_n$. (Unentangled elements are sometimes referred to as *simple tensors*.) Let $\Sigma(\mathcal{H}_1, \cdots, \mathcal{H}_n)$ be the set of all unentangled vectors of norm 1 in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. Every element in $\Sigma(\mathcal{H}_1, \cdots, \mathcal{H}_n)$ can be expressed as a tensor product of unit vectors in $\mathcal{H}_1, \cdots, \mathcal{H}_n$ respectively. Following Wallach [1], an *unentangled frame function* for $\mathcal{H}_1, \cdots, \mathcal{H}_n$ is a function $f : \Sigma(\mathcal{H}_1, \cdots, \mathcal{H}_n) \to \mathbb{R}^+$ such that, for some positive real number w (the *weight* of f) whenever $\{\xi_i\}_{i\in I}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with each $\xi_i \in \Sigma(\mathcal{H}_1, \cdots, \mathcal{H}_n)$, then $\sum_i f(\xi_i) = w$. The physical idea behind this definition is that the elements of $\Sigma(\mathcal{H}_1, \cdots, \mathcal{H}_n)$ represent the outcomes of elementary local operations or measurements.

It turns out that unentangled frame functions have natural links with quantum multi-measures. For the purposes of this note we define a (positive) *quantum multi-measure* for $\mathcal{H}_1, \cdots, \mathcal{H}_n$ to be a function $m : \mathcal{P}(\mathcal{H}_1) \times \cdots \times \mathcal{P}(\mathcal{H}_n) \to \mathbb{R}^+$, such that m is completely orthoadditive in each variable separately, see [5]. (Our results in [5] apply to more general, vector valued quantum multi-measures.) When $n = 2$, a multi-measure is called a *bi-measure*. These arise naturally in the study of quantum decoherence functionals [8,2–6].

Lemma III.1 *Let $\mathcal{H}_1, \cdots, \mathcal{H}_n$ be Hilbert spaces, none of which is of dimension 2. Let m be a (positive) quantum multi-measure for $\mathcal{H}_1, \cdots, \mathcal{H}_n$. Then there exists a unique bounded, multi-linear map $M : \mathcal{B}(\mathcal{H}_1) \times \cdots \times \mathcal{B}(\mathcal{H}_n) \to \mathbb{C}$, such that*

$$M(p_1, p_2, \cdots, p_n) = m(p_1, p_2, \cdots, p_n) \text{ for each } p_j \in \mathcal{B}(\mathcal{H}_j).$$

Furthermore, given r, with $1 \le r \le n$ and assuming $n \ge 2$, for each positive $x_j \in \mathcal{B}(\mathcal{H}_j)$, with $1 \le j \le n$ and $j \ne r$, the map $y \mapsto M(x_1, \cdots, x_{r-1}, y, x_{r+1}, \cdots, x_n)$ is a positive normal functional on $\mathcal{B}(\mathcal{H}_r)$.

Proof: The existence and uniqueness of M is a consequence of results obtained in [5]. Whenever x is a positive operator in $\mathcal{B}(\mathcal{H})$, there exists a sequence of commuting projections $\{p_j\}_{j=1,2,\cdots}$ such that $x = \|x\| \sum_j \frac{1}{2^j} p_j$ (for a proof see, e.g., [13] page 27). This observation, together with the positivity of m, shows that if x_r is positive for $r = 1, 2, \cdots, n$, then $M(x_1, \cdots, x_n) \ge 0$. It now follows from the results of [5] that given r, with $1 \le r \le n$ and assuming $n \ge 2$, for each positive $x_j \in \mathcal{B}(\mathcal{H}_j)$, with $1 \le j \le n$ and $j \ne r$, the map $y \mapsto M(x_1, \cdots, x_{r-1}, y, x_{r+1}, \cdots, x_n)$ is a positive normal functional on $\mathcal{B}(\mathcal{H}_r)$. $\square$

Let us recall that the algebraic tensor product $\mathcal{B}(\mathcal{H}_1) \otimes_{\mathrm{alg}} \cdots \otimes_{\mathrm{alg}} \mathcal{B}(\mathcal{H}_n)$ may be identified with the linear span of $\{x_1 \otimes x_2 \otimes \cdots \otimes x_n : x_j \in \mathcal{B}(\mathcal{H}_j)\}$ in (the von Neumann tensor product) $\mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n) =$

$\mathcal{B}(\mathcal{H}_1)\otimes\cdots\otimes\mathcal{B}(\mathcal{H}_n)$. Let m and M be as in Lemma III.1, then by the basic property of the algebraic tensor product, there exists a unique linear functional $\mathfrak{M}$ on $\mathcal{B}(\mathcal{H}_1)\otimes_{\text{alg}}\cdots\otimes_{\text{alg}}\mathcal{B}(\mathcal{H}_n)$ such that $\mathfrak{M}(x_1\otimes x_2\otimes\cdots\otimes x_n)=M(x_1,x_2,\cdots,x_n)$.

Corollary III.2 *Let $\mathcal{H}_1,\cdots,\mathcal{H}_n$ be finite dimensional Hilbert spaces, none of which has dimension 2. Let m be a positive quantum multi-measure for $\mathcal{H}_1,\cdots,\mathcal{H}_n$. Then there exists an unentangled frame function f for $\mathcal{H}_1,\cdots,\mathcal{H}_n$ such that, whenever $\nu_1\otimes\nu_2\otimes\cdots\otimes\nu_n$ is in $\Sigma(\mathcal{H}_1,\cdots,\mathcal{H}_n)$ and p_j is the projection of $\mathcal{H}_j$ onto the one-dimensional subspace generated by ν_j,*

$$f(\nu_1\otimes\nu_2\otimes\cdots\otimes\nu_n)=m(p_1,p_2,\cdots,p_n).$$

Proof: Fix a unit vector ν_j in $\mathcal{H}_j$ for $j=1,2,\cdots,n$. Then the projection from $\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n$ onto the subspace spanned by $\nu_1\otimes\nu_2\otimes\cdots\otimes\nu_n$ can be identified with the projection $p_1\otimes p_2\otimes\cdots\otimes p_n$ in $\mathcal{B}(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n)=\mathcal{B}(\mathcal{H}_1)\otimes\cdots\otimes\mathcal{B}(\mathcal{H}_n)$. Define $f(\nu_1\otimes\nu_2\otimes\cdots\otimes\nu_n)$ to be $\mathfrak{M}(p_1\otimes p_2\otimes\cdots\otimes p_n)=M(p_1,p_2,\cdots,p_n)=m(p_1,p_2,\cdots,p_n)$. $\square$

The following technical lemma allows us to associate a canonical multi-measure with each unentangled frame function.

Lemma III.3 *Let $\mathcal{H}_1,\cdots,\mathcal{H}_n$ be Hilbert spaces of arbitrary dimension and let $f:\Sigma(\mathcal{H}_1,\cdots,\mathcal{H}_n)\to\mathbb{R}^+$ be an unentangled frame function. Then there is a (positive, completely additive) quantum multi-measure m for $\mathcal{H}_1,\cdots,\mathcal{H}_n$ such that whenever $\nu_1\otimes\nu_2\otimes\cdots\otimes\nu_n$ is in $\Sigma(\mathcal{H}_1,\cdots,\mathcal{H}_n)$ and p_j is the projection of $\mathcal{H}_j$ onto the one-dimensional subspace generated by ν_j,*

$$m(p_1,p_2,\cdots,p_n)=f(\nu_1\otimes\nu_2\otimes\cdots\otimes\nu_n).$$

Proof: To simplify our notation we shall prove this for $n=2$, but the method is perfectly general.

Let e_1 and e_2 be projections in $\mathcal{P}(\mathcal{H}_1)$ and $\mathcal{P}(\mathcal{H}_2)$, respectively. Let E_1 and E_2 be the subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$ which are the respective ranges of e_1 and e_2. Let $\{\xi_j\}_{j\in J}$ and $\{\psi_i\}_{i\in I}$ be orthonormal bases of E_1 and E_2, respectively. We wish to define $m(e_1,e_2)$ to be

$$\sum_{j\in J}\sum_{i\in I}f(\xi_j\otimes\psi_i).$$

The only difficulty here is that we do not know that this number is independent of the choice of orthonormal bases for E_1 and E_2, respectively. To establish this we argue as follows.

Let w be the weight of f. Let $\{\xi_j\}_{j\in J^\perp}$ and $\{\psi_i\}_{i\in I^\perp}$ be orthonormal bases for $E_1^\perp$ and $E_2^\perp$, respectively. Then $\{\xi_j\otimes\psi_i\}_{j\in J\cup J^\perp,i\in I\cup I^\perp}$ is an orthonormal basis for $\mathcal{H}_1\otimes\mathcal{H}_2$. So

$$\sum_{(j,i)\in J\times I}f(\xi_j\otimes\psi_i)+\sum_{(j,i)\in J^\perp\times(I\cup I^\perp)}f(\xi_j\otimes\psi_i)+\sum_{(j,i)\in(J\cup J^\perp)\times I^\perp}f(\xi_j\otimes\psi_i)=w.$$

Let $\{\xi'_j\}_{j\in J}$ and $\{\psi'_i\}_{i\in I}$ be orthonormal bases of E_1 and E_2, respectively. Then

$$\sum_{(j,i)\in J\times I}f(\xi'_j\otimes\psi'_i)+\sum_{(j,i)\in J^\perp\times(I\cup I^\perp)}f(\xi_j\otimes\psi_i)+\sum_{(j,i)\in(J\cup J^\perp)\times I^\perp}f(\xi_j\otimes\psi_i)=w.$$

Hence

$$\begin{aligned}\sum_{(j,i)\in J\times I} f(\xi'_j\otimes\psi'_i) &= w - \sum_{(j,i)\in J^{\perp}\times(I\cup I^{\perp})} f(\xi_j\otimes\psi_i) - \sum_{(j,i)\in(J\cup J^{\perp})\times I^{\perp}} f(\xi_j\otimes\psi_i)\\ &= \sum_{(j,i)\in J\times I} f(\xi_j\otimes\psi_i).\end{aligned}$$

So m is well-defined. It is straightforward to verify that m has all the required properties. □

Remark: In the above argument we made essential use of the property that f is an unentangled frame function. Suppose that we only knew that f satisfied the weaker property: for some positive real number w whenever $\{\xi_i\}_{i\in I}$ is a product orthonormal basis of $\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n$, then $\sum_i f(\xi_i)=w$. Then the proof of the preceding lemma would break down. This throws fresh light on the counterexample constructed in Proposition 5 in [1].

In our investigations on quantum decoherence functionals we were led to obtain results on generalised quantum bi-measures and multi-measures [2–6]. The statement of the next theorem is Wallach's Theorem 1 [1]. Our proof shows that Wallach's Theorem is a natural consequence of our earlier results on quantum multi-measures.

Proposition III.4 (Wallach, Theorem 1 [1]) *Let $\mathcal{H}_1,\cdots,\mathcal{H}_n$ be finite dimensional Hilbert spaces, each of dimension at least 3. Let $f:\Sigma(\mathcal{H}_1,\cdots,\mathcal{H}_n)\to\mathbb{R}^+$ be an unentangled frame function. Then there exists a self-adjoint operator T in $\mathcal{B}(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n)$ such that whenever $\nu_1\otimes\nu_2\otimes\cdots\otimes\nu_n$ is in $\Sigma(\mathcal{H}_1,\cdots,\mathcal{H}_n)$ and p_j is the projection of $\mathcal{H}_j$ onto the one-dimensional subspace generated by ν_j,*

$$f(\nu_1\otimes\nu_2\otimes\cdots\otimes\nu_n)=\mathrm{Tr}((p_1\otimes p_2\otimes\cdots\otimes p_n)T).$$

Proof: Since each of the Hilbert spaces $\mathcal{H}_1,\cdots,\mathcal{H}_n$ is finite dimensional, $\mathcal{B}(\mathcal{H}_1)\otimes\cdots\otimes\mathcal{B}(\mathcal{H}_n)=\mathcal{B}(\mathcal{H}_1)\otimes_{\mathrm{alg}}\cdots\otimes_{\mathrm{alg}}\mathcal{B}(\mathcal{H}_n)$. Let m be the quantum multi-measure constructed from f as in Lemma III.3. Let $\mathfrak{M}$ be the linear functional on $\mathcal{B}(\mathcal{H}_1)\otimes_{\mathrm{alg}}\cdots\otimes_{\mathrm{alg}}\mathcal{B}(\mathcal{H}_n)=\mathcal{B}(\mathcal{H}_1)\otimes\cdots\otimes\mathcal{B}(\mathcal{H}_n)=\mathcal{B}(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n)$ such that $\mathfrak{M}(q_1\otimes q_2\otimes\cdots\otimes q_n)=M(q_1,q_2,\cdots,q_n)=m(q_1,q_2,\cdots,q_n)$ for each $q_j\in\mathcal{P}(\mathcal{H}_j)$. Since $\mathfrak{M}$ is a linear functional on a finite dimensional space, it is bounded. Hence there is a unique bounded operator T in $\mathcal{B}(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n)$ such that $\mathfrak{M}(x)=\mathrm{Tr}(xT)$ for all x. Thus

$$f(\nu_1\otimes\nu_2\otimes\cdots\otimes\nu_n)=\mathfrak{M}(p_1\otimes p_2\otimes\cdots\otimes p_n)=\mathrm{Tr}((p_1\otimes p_2\otimes\cdots\otimes p_n)T). \tag{1}$$

On taking complex conjugates of the Equation (1) we find that T may be replaced by T^*. So in (1) we may replace T by $\frac{1}{2}(T+T^*)$. Hence we may suppose in (1) that T is self-adjoint. □

The work of [5,6] shows that Wallach's Theorem can be generalised to the situation where the Hilbert spaces are not required to be finite dimensional provided an appropriate boundedness condition is imposed. More precisely:

Theorem III.5 *Let $\mathcal{H}_1, \cdots, \mathcal{H}_n$ be Hilbert spaces, each of dimension at least 3. Let $f : \Sigma(\mathcal{H}_1, \cdots, \mathcal{H}_n) \to \mathbb{R}^+$ be an unentangled frame function. Let $\mathfrak{M}$ be the associated linear functional on $\mathcal{B}(\mathcal{H}_1) \otimes_{\text{alg}} \cdots \otimes_{\text{alg}} \mathcal{B}(\mathcal{H}_n)$. If the restriction of $\mathfrak{M}$ to $\mathcal{K}(\mathcal{H}_1) \otimes_{\text{alg}} \cdots \otimes_{\text{alg}} \mathcal{K}(\mathcal{H}_n)$ is bounded, then there exists a unique bounded self-adjoint, trace class operator T in $\mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ such that whenever $\nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n$ is in $\Sigma(\mathcal{H}_1, \cdots, \mathcal{H}_n)$ and p_j is the projection of $\mathcal{H}_j$ onto the one-dimensional subspace generated by ν_j,*

$$f(\nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n) = \mathrm{Tr}((p_1 \otimes p_2 \otimes \cdots \otimes p_n)T).$$

Proof: Let $\mathfrak{M}_0$ be the restriction of $\mathfrak{M}$ to $\mathcal{K}(\mathcal{H}_1) \otimes_{\text{alg}} \cdots \otimes_{\text{alg}} \mathcal{K}(\mathcal{H}_n)$. By hypothesis $\mathfrak{M}_0$ is bounded and so has a unique bounded extension, also denoted by $\mathfrak{M}_0$, to $\mathcal{K}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$. By standard functional analysis, there exists a trace class operator T such that $\mathfrak{M}_0(z) = \mathrm{Tr}(zT)$ for each z in $\mathcal{K}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$. Since each one-dimensional projection in $\mathcal{B}(\mathcal{H}_j)$ is in $\mathcal{K}(\mathcal{H}_j)$,

$$m(p_1, p_2, \cdots, p_n) = M(p_1, p_2, \cdots, p_n) = \mathfrak{M}_0(p_1 \otimes p_2 \otimes \cdots \otimes p_n) = \mathrm{Tr}((p_1 \otimes p_2 \otimes \cdots \otimes p_n)T).$$

So

$$f(\nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n) = \mathrm{Tr}((p_1 \otimes p_2 \otimes \cdots \otimes p_n)T) = \langle T(\nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n), \nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. It now follows from Lemma 5.9 [6] that T is unique. But, arguing as in the proof of Proposition III.4, we can replace T by $\frac{1}{2}(T + T^*)$. So, by uniqueness, T is self-adjoint. $\square$

Remark: When the spaces $\mathcal{H}_1, \cdots, \mathcal{H}_n$ are finite dimensional, then the boundedness condition of Theorem III.5 is automatically satisfied. So Wallach's Theorem is a corollary of Theorem III.5 which, in turn, follows from the work of [5,6].

Moreover, it can be shown along the lines of [3] that, for $n = 2$, there exists a self-adjoint operator T, not necessarily of trace class, on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, such that

$$\mathfrak{M}(p) = \mathrm{Tr}(Tp)$$

for all finite rank projections p in $\mathcal{P}(\mathcal{H})$ if, and only if, $\mathfrak{M}$ is bounded on one dimensional projection operators whose ranges are generated by vectors in the algebraic tensor product $\mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2$.

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